Solving with Absolute Value

Who knew two little lines could cause so much trouble? Ask someone to solve the equation $3x - 2 = 7$ and they'll say "No problem!" Add just two little lines, and ask them to solve the equation $|3x-2| = 7$ and you're likely to get a blank stare. Unless, of course, they happen to have taken PreCalculus at UConn.

Most students have seen absolute value before and learned of it as something that "makes things positive." That's one good way of viewing it. The absolute value of -5 is 5, or in other words, taking the absolute value of -5 "makes it positive." If something's already positive, then absolute value signs do nothing to it.

$$|-5| = 5 \qquad\qquad |5| = 5$$

Now, where things get confusing is if we *don't know* if the thing inside absolute value signs is positive or negative. This situation comes up when we need to solve equations or inequalities that involve absolute value. For example, if I had $|3x - 2| = 7$, well, I don't know if $3x - 2$ is a positive number or a negative number because I don't yet know what x is. So how can I simplify that?

In order to understand this, we need a more sophisticated view of what the absolute value of a quantity is. Let's write it this way:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Now, let's decipher that. What we're saying here is that the absolute value of some quantity x is just x, *if x is already positive.* (Or if $x = 0$...so we can just say, "if x is nonnegative.") On the other hand, *if x is negative*, then the absolute value of x is $-x$. That corresponds to our idea of "absolute value makes things positive" because if x is negative, then $-x$ is a positive number. Try it yourself, with $x = -3$. In that case $|-3| = -(-3) = 3$.

This new way of looking at absolute value can be used to solve equations like the one above, $|3x-2| = 7$. If we see this equation, we know that either $|3x-2| = 3x-2$ (if $3x-2$ is a positive quantity), or $|3x-2| = -(3x-2)$ (if $3x-2$ is a negative quantity). So we can handle each case individually.

(1) If $3x - 2$ is positive, then $|3x - 2| = 7$ simplifies to $3x - 2 = 7$, so we can add 2 to both sides to get $3x = 9$ and divide both sides by 3 to get $x = 3$.
(2) If $3x-2$ is negative, then $|3x-2| = 7$ simplifies to $-(3x-2) = 7$, so we can distribute the negative to get $-3x + 2 = 7$, subtract 2 from both sides to get $-3x = 5$, and divide both sides by -3 to get $x = -\frac{5}{3}$.

Therefore, we have two possible solutions, $x = 3$, or $x = -\frac{5}{3}$. That is, if we plug in either of these numbers for x into the original equation, we get something true.

Try a few similar problems yourself. Note: one of these is a "trick" question – find it and explain why.

(1) $|x - 2| = 7$

(2) $|7 - x| = 10$

(3) $8 - |x+1| = 5$

(4) $|3-2x| + 4 = 1$

(5) Question: is $|x|$ the same as $|-x|$?

(6) Question: is $|3x-2|$ the same as $|2-3x|$?

(7) Question: is $|3x-2|$ the same as $-|3x-2|$?

We can use this same big idea, *break it up into cases*, so solve inequalities that involve absolute value. For example, say we want to solve the inequality $|x-4| > 5$. Let's break it into our two cases.

- Let's suppose that $x-4 \geq 0$ (which is the same as saying that $x \geq 4$). Then $|x-4| = x-4$, so our inequality becomes $x-4 > 5$, which simplifies to $x > 9$.
- Let's suppose now that $x-4 < 0$ (i.e., $x < 4$). Then $|x-4| = -(x-4)$, so our inequality becomes $-(x-4) > 5$, which simplifies to $-x+4 > 5$, which simplifies to $-x > 1$, which simplifies to $x < -1$.

Therefore, our solution set to this inequality, the set of all values for x that makes the inequality true, is $x > 9$ or $x < -1$. In interval notation, this is $(-\infty, -1) \cup (9, \infty)$.

Let's do another example. Let's say we want to solve the inequality $|3x+1| - 2 \leq 5$. We could simplify this first, before dealing with the absolute value: add 2 to both sides and we get: $|3x+1| \leq 7$. Now, what we're concerned about is when $3x+1$ is nonnegative and when $3x+1$ is negative. Let's figure out exactly where that happens: it's nonnegative when $3x+1 \geq 0$, which is when $3x \geq -1$, which is when $x \geq -\frac{1}{3}$. Otherwise, it must be negative. So we have our two cases:

- If $x \geq -\frac{1}{3}$, then $|3x+1| \leq 7$ is the same as $3x+1 \leq 7$, which we can solve:

$$\begin{aligned} 3x+1 &\leq 7 \\ 3x &\leq 6 \\ x &\leq 2 \end{aligned}$$

- If $x < -\frac{1}{3}$, then $|3x+1| \leq 7$ is the same as $-(3x+1) \leq 7$, which we can solve:

$$\begin{aligned} -(3x+1) &\leq 7 \\ 3x+1 &\geq -7 \\ 3x &\geq -8 \\ x &\geq -\frac{8}{3} \end{aligned}$$

So our solution set, all values that make this equation true, is $[-\frac{8}{3}, 2]$. Now your turn to try a few.

(8) $|x+7| \geq 15$

(9) $|x - \frac{1}{2}| < 5$

(10) $|2x-8| \leq 10$

(11) $|5x+1| > 1$

Now you have the knowledge, believe it or not, to solve even more complicated things like $|x+1|+|x-2| < 5$. Remember, when there's a variable inside an absolute value sign, you can break it up into cases: when the thing inside the absolute value is positive (or nonnegative) and when it's negative. The factor $(x+1)$ is nonnegative when $x \geq -1$, and negative when $x < -1$. The factor $(x-2)$ is nonnegative when $x \geq 2$, and negative when $x < 2$. So, if we want to consider all possible cases, we have:

- $x < -1$. In this case, both things inside the absolute value are negative, so $|x+1| = -(x+1)$ and $|x-2| = -(x-2)$.
- $-1 \leq x < 2$. In this case, one is negative and one is positive, i.e., $|x+1| = x+1$ and $|x-2| = -(x-2)$.
- $x \geq 2$. In this case, both are positive, so $|x+1| = x+1$ and $|x-2| = x-2$.

So to solve this one, we just have to divide up into those cases:

- $x < -1$. Then our inequality becomes $-(x+1) - (x-2) < 5$, which we can solve:

$$\begin{aligned} -(x+1) - (x-2) &< 5 \\ -x - 1 - x + 2 &< 5 \\ -2x + 1 &< 5 \\ -2x &< 4 \\ x &> -2 \end{aligned}$$

  So if $x < -1$, then the inequality is true if $x > -2$. So we have some solutions, $(-2, -1)$.
- $-1 \leq x < 2$. Then the inequality becomes $(x+1) - (x-2) < 5$:

$$\begin{aligned} (x+1) - (x-2) &< 5 \\ x + 1 - x + 2 &< 5 \\ 3 &< 5 \end{aligned}$$

  This is always true, no matter what x is! So, if x is any value between -1 and 2, the inequality is true. So we have more solutions: $[-1, 2)$.
- $x \geq 2$. Then the inequality becomes $(x+1) + (x-2) < 5$:

$$\begin{aligned} (x+1) + (x-2) &< 5 \\ 2x - 1 &< 5 \\ 2x &< 6 \\ x &< 3 \end{aligned}$$

  So if $x \geq 2$, then the inequality is true if $x < 3$. So our remaining solutions are $[2, 3)$.

Put all of these cases together, and all the values of x that make the inequality true are $(-2,-1) \cup [-1,2) \cup [2,3)$, which is just the same as the single interval $(-2, 3)$.

Your turn!

(12) $|x+3|-|x-2|<1$

(13) $|x-5|+|x-7|\geq 10$

(14) $|2x-1|+|x|>4$